

ELEMENTS OF GENERAL RELATIVITY

It is not the purpose of this chapter to introduce or explain Einstein's general theory of relativity, since the reader who is not already familiar with it is unlikely to gain much from this book. The main purpose here is simply to clarify the notation that will be used. It is also appropriate in this chapter to briefly introduce the Newman–Penrose formalism which facilitates the geometrical analysis of the colliding plane wave problem and which will be used in Chapter 6 to derive the field equations.

2.1 Basic notation

Basically, we will be following a very traditional approach, and the notation adopted will be that of the well known paper of Newman and Penrose (1962).

Accordingly, a space-time will be represented by a connected C^∞ Hausdorff manifold M together with a locally Lorentz metric $g_{\mu\nu}$ with signature $(+, -, -, -)$ and a symmetric linear connection $\Gamma^\lambda_{\mu\nu}$. Greek indices are used to indicate the values 0,1,2,3, and the covariant derivative of a vector is given by

$$A^\lambda_{;\nu} = A^\lambda_{,\nu} + \Gamma^\lambda_{\mu\nu} A^\mu \quad (2.1)$$

where a comma denotes a partial derivative.

The curvature tensor is given in terms of the connection by

$$R^\lambda_{\kappa\mu\nu} = \Gamma^\lambda_{\kappa\nu,\mu} - \Gamma^\lambda_{\kappa\mu,\nu} + \Gamma^\lambda_{\alpha\mu} \Gamma^\alpha_{\kappa\nu} - \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\kappa\mu}. \quad (2.2)$$

The Ricci tensor, which is the first contraction of the curvature tensor, is given by

$$R_{\mu\nu} = -R^\alpha_{\mu\alpha\nu}. \quad (2.3)$$

The curvature tensor has twenty independent components. These can be considered as the ten independent components of the Ricci tensor, and the ten independent components of the Weyl tensor, which is the trace free part of the curvature tensor, and is given by

$$\begin{aligned} C_{\kappa\lambda\mu\nu} = & R_{\kappa\lambda\mu\nu} - \frac{1}{2}(R_{\lambda\mu}g_{\kappa\nu} - R_{\lambda\nu}g_{\kappa\mu} - R_{\kappa\mu}g_{\lambda\nu} + R_{\kappa\nu}g_{\lambda\mu}) \\ & + \frac{1}{6}R(g_{\lambda\mu}g_{\kappa\nu} - g_{\lambda\nu}g_{\kappa\mu}) \end{aligned} \quad (2.4)$$

where $R = R_\alpha^\alpha$ is the curvature scalar.

These two groups of components have different physical interpretations. The components of the Ricci tensor are related to the energy-momentum tensor $T_{\mu\nu}$ of the matter field present, through Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}. \quad (2.5)$$

These components can be considered to define the amount of curvature that is directly generated by the matter fields that are present at any location. For a vacuum field they will be zero, but they will be non-zero when electromagnetic waves or other fields are present.

The components of the Weyl tensor, on the other hand, define the 'free gravitational field'. They may be considered as describing the components of curvature that are not generated locally. In this sense they describe the pure gravitational field components. They may be interpreted as the components of gravitational waves, or of gravitational fields generated by non-local sources.

2.2 Components of the curvature tensor

It is convenient to represent the curvature tensor in terms of distinct sets of components. Not only may it be divided into the Weyl and Ricci tensors, but each of these tensors may be described in terms of distinct components. The appropriate notation here is that of Newman and Penrose (1962).

It is found to be convenient to introduce a tetrad system of null vectors. These include two real null vectors l^μ and n^μ , a complex null vector m^μ , and its conjugate. They are defined such that their only non-zero inner products are

$$l_\mu n^\mu = 1, \quad m_\mu \bar{m}^\mu = -1, \quad (2.6)$$

and they must satisfy the completeness relation

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \quad (2.7)$$

Having defined a tetrad basis, the Ricci and Weyl tensors may now be expressed in terms of their tetrad components. The ten independent components of the Ricci tensor can conveniently be divided into a component Λ representing the curvature scalar and the nine independent components of a Hermitian 3×3 matrix Φ_{AB} which represents the trace free part of the Ricci tensor and satisfies

$$\Phi_{AB} = \bar{\Phi}_{BA} \quad (2.8)$$

where $A, B = 0, 1, 2$. These components are defined by

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu \\
\Phi_{01} &= -\frac{1}{2}R_{\mu\nu}l^\mu m^\nu \\
\Phi_{02} &= -\frac{1}{2}R_{\mu\nu}m^\mu m^\nu \\
\Phi_{11} &= -\frac{1}{4}R_{\mu\nu}(l^\mu n^\nu + m^\mu \bar{m}^\nu) \\
\Phi_{12} &= -\frac{1}{2}R_{\mu\nu}n^\mu m^\nu \\
\Phi_{22} &= -\frac{1}{2}R_{\mu\nu}n^\mu n^\nu \\
\Lambda &= \frac{1}{24}R.
\end{aligned} \tag{2.9}$$

The ten independent components of the Weyl tensor, representing the free gravitational field, can more conveniently be expressed as the five complex scalars

$$\begin{aligned}
\Psi_0 &= -C_{\kappa\lambda\mu\nu}l^\kappa m^\lambda l^\mu m^\nu \\
\Psi_1 &= -C_{\kappa\lambda\mu\nu}l^\kappa n^\lambda l^\mu m^\nu \\
\Psi_2 &= -C_{\kappa\lambda\mu\nu}l^\kappa m^\lambda \bar{m}^\mu n^\nu \\
\Psi_3 &= -C_{\kappa\lambda\mu\nu}n^\kappa l^\lambda n^\mu \bar{m}^\nu \\
\Psi_4 &= -C_{\kappa\lambda\mu\nu}n^\kappa \bar{m}^\lambda n^\mu \bar{m}^\nu.
\end{aligned} \tag{2.10}$$

These components have distinct physical interpretations that will be mentioned below. They also have particular convenience when considering the algebraic classification of the space-time.

Gravitational fields are usually classified according to the Petrov–Penrose classification of the Weyl tensor. This is based on the number of its distinct principal null directions and the number of times these are repeated. This classification is most conveniently described using a spinor approach. However, there is no need to introduce spinors here, as tetrads are sufficient.

According to the tetrad approach, a null vector k^μ is said to describe a principal null direction of the gravitational field with multiplicity 1, 2, 3 or 4 if it satisfies respectively

$$\begin{aligned}
k_{[\rho}C_{\kappa]\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu &= 0 \\
C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu &= 0 \\
C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^\mu &= 0 \\
C_{\kappa\lambda\mu\nu}k^\mu &= 0
\end{aligned} \tag{2.11}$$

where square brackets are used to denote the antisymmetric part. There are at most four principal null directions.

If all four principal null directions are distinct, the space-time is said to be algebraically general, or of type I. If there is a repeated principal null direction, then the space-time is said to be algebraically special. If it has multiplicity two, three or four, the space-time is said to be of types II, III or N respectively. If a space-time has two distinct repeated principal null directions, it is said to be of type D. If the Weyl tensor is zero, the space-time is conformally flat or of type O.

If either of the basis vectors l^μ or n^μ are aligned with principal null directions then either $\Psi_0 = 0$ or $\Psi_4 = 0$ respectively. If the vector l^μ is aligned with the repeated principal null direction of an algebraically special space-time, then $\Psi_0 = \Psi_1 = 0$. If this principal null direction is repeated two, three or four times, then the only non-zero components of the Weyl tensor are the sets (Ψ_2, Ψ_3, Ψ_4) , (Ψ_3, Ψ_4) or Ψ_4 respectively. Finally, if l^μ and n^μ are both aligned with the distinct principal null directions of a type D space-time, then the only non-zero component of the Weyl tensor is Ψ_2 .

The physical meaning of the different components of the Weyl tensor has been investigated by Szekeres (1965), and may be summarized as follows:

- Ψ_0 denotes a transverse wave component in the n^μ direction.
- Ψ_1 denotes a longitudinal wave component in the n^μ direction.
- Ψ_2 denotes a coulomb component.
- Ψ_3 denotes a longitudinal wave component in the l^μ direction.
- Ψ_4 denotes a transverse wave component in the l^μ direction.

This interpretation will be very useful when we come to analyse the interaction between two gravitational waves. It is possible to align the two basis vectors l^μ and n^μ with the two waves. The interaction to be considered here is between transverse waves, so the problem is to find the interaction between the Ψ_4 and Ψ_0 components.

2.3 Spin coefficients

In some situations it has been found convenient to modify the notation for tetrads, spinors and spin coefficients in recent years.¹ However, it is most convenient here to continue to use the original notation of Newman and Penrose (1962).² Those not familiar with this notation need not be

¹ See Geroch, Held and Penrose (1973), Penrose and Rindler (1985).

² For a detailed introduction to the Newman–Penrose formalism see Pirani (1965), Carmeli (1977), Alekseev and Klebnikov (1978), Frolov (1979) and Kramer *et al.* (1980).

too alarmed, as we have no need here to introduce spinors or their more intricate properties. Effectively, we are only introducing the notation because of its convenience for describing certain geometrical properties of the solutions.

The main feature of the Newman–Penrose formalism is the introduction of spin coefficients. These are complex linear combinations of the Ricci rotation coefficients associated with the null tetrad. They are defined by

$$\begin{aligned}
\kappa &= l_{\mu;\nu} m^\mu l^\nu & \nu &= -n_{\mu;\nu} \bar{m}^\mu n^\nu \\
\rho &= l_{\mu;\nu} m^\mu \bar{m}^\nu & \mu &= -n_{\mu;\nu} \bar{m}^\mu m^\nu \\
\sigma &= l_{\mu;\nu} m^\mu m^\nu & \lambda &= -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu \\
\tau &= l_{\mu;\nu} m^\mu n^\nu & \pi &= -n_{\mu;\nu} \bar{m}^\mu l^\nu \\
\epsilon &= \frac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu) \\
\alpha &= \frac{1}{2}(l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu) \\
\beta &= \frac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu) \\
\gamma &= \frac{1}{2}(l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu).
\end{aligned} \tag{2.12}$$

The spin coefficients have the following geometrical interpretations. If $\kappa = 0$, then l^μ is tangent to a geodesic null congruence. If, in addition, $\mathcal{R}e \epsilon = 0$, then l^μ is the tangent vector corresponding to an affine parametrization, and $-\mathcal{R}e \rho$, $\mathcal{I}m \rho$ and $|\sigma|$ define the expansion, twist and shear of the congruence respectively. Also, $\arg \sigma$ determines the shear axes. With a change of sign, a positive value for $\mathcal{R}e \rho$ is more appropriately referred to as the contraction of the congruence.

It may also be noted that l_μ is proportional to the gradient of a scalar field if, and only if, it is tangent to a twist-free null geodesic congruence ($\kappa = 0$, $\rho - \bar{\rho} = 0$). Also, l_μ is equal to the gradient of a scalar field when $\kappa = 0$, $\rho - \bar{\rho} = 0$, $\epsilon + \bar{\epsilon} = 0$ and $\bar{\alpha} + \beta = \tau$.

In the congruence defined by n^μ the coefficients $-\nu$, $-\gamma$, $-\mu$, $-\lambda$ correspond to κ , ϵ , ρ , σ respectively. The geometrical properties of the solutions given later can be very conveniently analysed in terms of these spin coefficients.

We also need to define the intrinsic derivatives. These are directional derivatives in the directions of the four tetrad vectors, and are defined by

$$D = l^\mu \nabla_\mu, \quad \Delta = n^\mu \nabla_\mu, \quad \delta = m^\mu \nabla_\mu, \quad \bar{\delta} = \bar{m}^\mu \nabla_\mu \tag{2.13}$$

where ∇_μ is the covariant derivative operator, previously denoted by a semicolon.

Basic to their formalism is the set of Newman–Penrose identities. The first group of these are linear combinations of the Ricci identities applied to the tetrad vectors. These will only be referred to occasionally in the following sections, but it is still worth quoting them here as follows:

$$D\rho - \bar{\delta}\kappa = \rho^2 + \sigma\bar{\sigma} + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00} \quad (2.14a)$$

$$D\sigma - \delta\kappa = \sigma(\rho + \bar{\rho}) + \sigma(3\epsilon - \bar{\epsilon}) - \kappa(\pi - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0 \quad (2.14b)$$

$$D\tau - \Delta\kappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \kappa(3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01} \quad (2.14c)$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\rho + \epsilon) + \Phi_{10} \quad (2.14d)$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1 \quad (2.14e)$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11} \quad (2.14f)$$

$$D\lambda - \bar{\delta}\pi = \rho\lambda + \bar{\sigma}\mu + \pi^2 + \pi(\alpha - \bar{\beta}) - \nu\bar{\kappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20} \quad (2.14g)$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \sigma\lambda + \pi\bar{\pi} - \mu(\epsilon + \bar{\epsilon}) - \pi(\bar{\alpha} - \beta) - \nu\kappa + \Psi_2 + 2\Lambda \quad (2.14h)$$

$$D\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) + \Psi_3 + \Phi_{21} \quad (2.14i)$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu}) - \lambda(3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4 \quad (2.14j)$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01} \quad (2.14k)$$

$$\delta\alpha - \bar{\delta}\beta = \rho\mu - \sigma\lambda + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Lambda + \Phi_{11} \quad (2.14l)$$

$$\delta\lambda - \bar{\delta}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21} \quad (2.14m)$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22} \quad (2.14n)$$

$$\delta\gamma - \Delta\beta = \tau(\mu + \gamma) - \gamma\bar{\alpha} - \sigma\nu - \epsilon\bar{\nu} - \beta(2\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12} \quad (2.14o)$$

$$\delta\tau - \Delta\sigma = \mu\sigma + \rho\bar{\lambda} + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \kappa\bar{\nu} + \Phi_{02} \quad (2.14p)$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \sigma\lambda + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \kappa\nu - \Psi_2 - 2\Lambda \quad (2.14q)$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3. \quad (2.14r)$$

We will also require the commutation relations between the intrinsic derivatives. When applied to scalar functions, these are given by

$$\begin{aligned} \Delta D - D\Delta &= (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta \\ \delta D - D\delta &= (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta \\ \delta\Delta - \Delta\delta &= -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta \\ \bar{\delta}\delta - \delta\bar{\delta} &= (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\beta - \bar{\alpha})\bar{\delta} + (\alpha - \bar{\beta})\delta. \end{aligned} \quad (2.15)$$

These can be applied to the coordinates to give the so-called metric equations, and also to the spin coefficients and the curvature tensor components.

An essential part of the general Newman–Penrose formalism is the set of Bianchi identities. These, however, will not be used explicitly in this text and therefore do not need to be repeated here.

2.4 Einstein–Maxwell fields

In this formalism, Einstein’s field equations are applied simply by replacing the expressions for the Ricci tensor components in the above identities by the appropriate components of the energy-momentum tensor according to equation (2.5). For an electromagnetic field in a vacuum, for example, it is convenient to represent the electromagnetic field tensor $F^{\mu\nu}$ by three complex scalars defined by

$$\begin{aligned}\Phi_0 &= F_{\mu\nu} l^\mu m^\nu \\ \Phi_1 &= \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + \bar{m}^\mu m^\nu) \\ \Phi_2 &= F_{\mu\nu} \bar{m}^\mu n^\nu.\end{aligned}\tag{2.16}$$

It is also possible to scale the electromagnetic field tensor such that Einstein’s gravitational field equations are then given by

$$\Phi_{AB} = \Phi_A \bar{\Phi}_B, \quad \Lambda = 0,\tag{2.17}$$

and, in this case, Maxwell’s equations take the form

$$\begin{aligned}D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \\ D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \\ \delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \\ \delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2\end{aligned}\tag{2.18}$$

It is also possible to classify the electromagnetic field in a similar way to the classification of the Weyl tensor. An electromagnetic field is said to be non-null or null if it has two distinct or one repeated principal null direction k^μ satisfying respectively

$$\begin{aligned}F_{\mu[\nu} k_{\lambda]} k^\mu &= 0 \\ F_{\mu\nu} k^\mu &= 0, \quad F_{[\mu\nu} k_{\lambda]} = 0.\end{aligned}\tag{2.19}$$

Aligning l^μ with a principal null direction of the electromagnetic field makes $\Phi_0 = 0$. If l^μ is a repeated principal null direction, then $\Phi_0 = \Phi_1 = 0$, and the only non-zero component of the null field is Φ_2 .